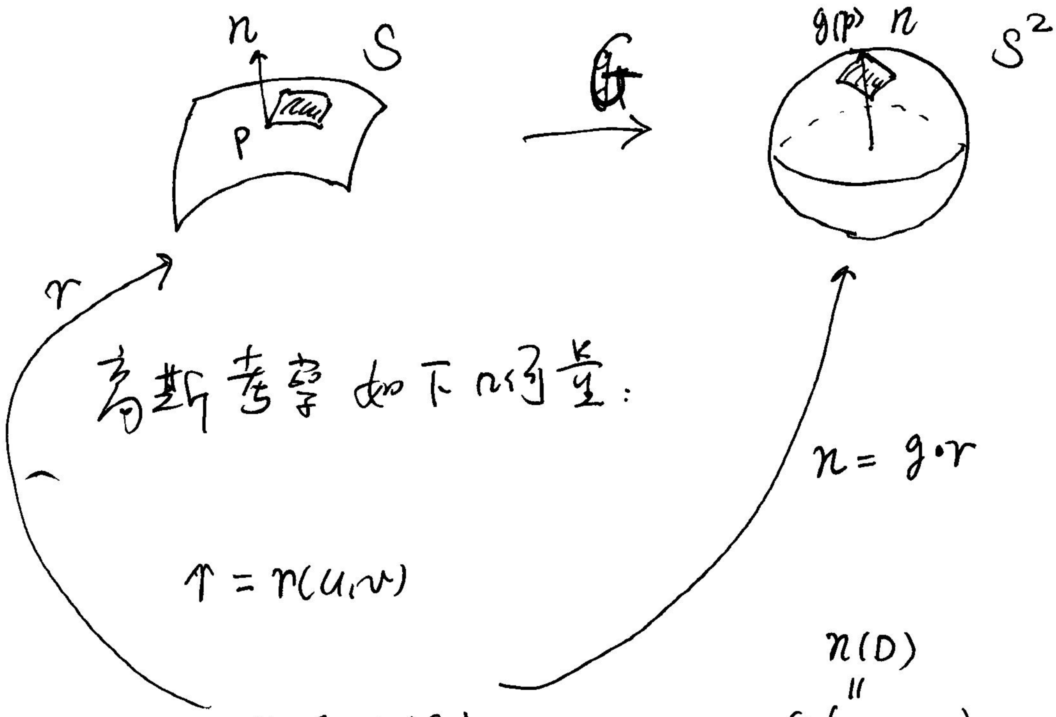
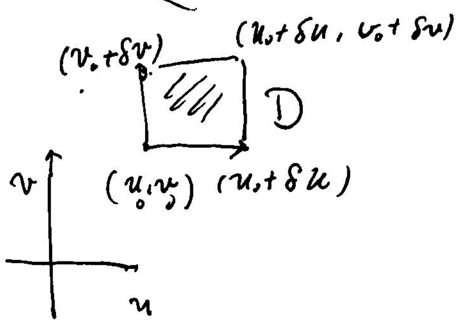
§6. 高斯曲率

高斯映射:



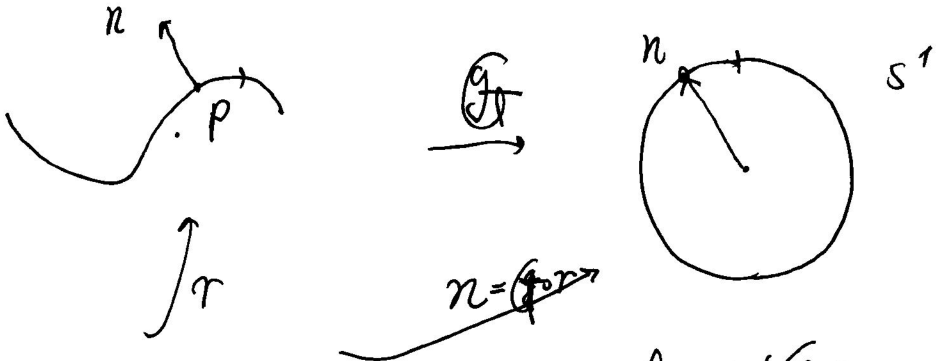
高斯考察如下的量:

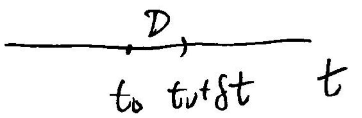


$$\lim_{\substack{\delta u\to 0\\ \delta v\to 0}} \frac{\mathrm{Area}(G(r(D)))}{\mathrm{Area}(r(D))} \triangleq \text{高斯曲率}$$

$$n(D) = G(r(D))$$

回忆: 设 $r = r(t)$ 平面曲线





$$\lim_{\delta t\to 0} \frac{\mathrm{length}(G(r(D)))}{\mathrm{length}(r(D))} = \kappa(p)$$

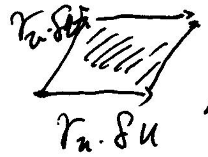
先考察<u>简单情形</u>:

$$z = f(u,v) \qquad (\text{图}).$$

$$r(u,v) = (u, v, f(u,v))$$

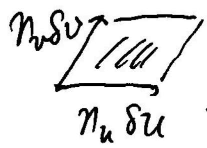
$$r_u = (1, 0, f_u), \quad r_v = (0, 1, f_v)$$

$$\Rightarrow n = \frac{r_u \times r_v}{|r_u \times r_v|} = \frac{1}{\sqrt{1+f_u^2+f_v^2}}(-f_u, -f_v, 1)$$

$\mathrm{Area}(r(D)) \approx \mathrm{Area}\Bigg($  $\Bigg)$

$$\| $$

$$|r_u \times r_v|\, \delta u \cdot \delta v = \sqrt{1+f_u^2+f_v^2}\, \delta u \cdot \delta v$$

$\mathrm{Area}(g(r(D))) \approx \mathrm{Area}\Bigg($  $\Bigg)$

$$\|$$

$$|n_u \times n_v|\, \delta u \cdot \delta v$$

$$n_u = (1+f_u^2+f_v^2)^{-3/2}\Big(-f_{uu}(1+f_v^2)+f_uf_vf_{uv},$$
$$-f_{uv}(1+f_u^2)+f_uf_vf_{uu},$$
$$-(f_uf_{uu}+f_vf_{uv})\Big)$$

$$n_v = (1+f_u^2+f_v^2)^{-3/2}\Big(-f_{uv}(1+f_v^2)+f_uf_vf_{vv},$$
$$-f_{vv}(1+f_u^2)+f_uf_vf_{uv},$$
$$-(f_vf_{vv}+f_uf_{uv})\Big)$$

$$n_u \times n_v = \left[\frac{(f_{uu}f_{vv}-f_{uv}^2)}{(1+f_u^2+f_v^2)^2}\right](f_u, f_v, -1)$$

$$(-f_u, -f_v, 1)$$

注意到 ~~note that~~: $n_u \times n_v \parallel r_u \times r_v$.

这是因为 S^2 在点 n 处的外法向量即为 n 向量本身.

而 $\{n_u, n_v\}$ 为 S^2 在点 n 处的切平面的一组基, 故

$n_u \times n_v \parallel n \parallel r_u \times r_v$. 故得结论.

在以后，我们将考察 线性映射

因为什么?!

$$(dG)_p : T_pS \longrightarrow T_{G(p)}S^2 = T_pS$$

$$\{r_u, r_v\} \longmapsto \{n_u, n_v\}$$

所以

$$K(p) = \lim_{\substack{\delta u, \delta v \\ \to 0}} \frac{\text{Area}(n(D))}{\text{Area}(r(D))} = \frac{|n_u \times n_v|}{|r_u \times r_v|}$$

$$= \frac{f_{uu} f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2} \cdot \cancelto{1}{\frac{|r_u \times r_v|}{|r_u \times r_v|}}$$

另一方面，我们容易计算到

$$\mathrm{I}(u,v) = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix}$$

$$\mathrm{II}(u,v) = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

故我们观察到等式：

$$\kappa(p) = \frac{\det \mathrm{II}(p)}{\det \mathrm{I}(p)}$$

（或 $\kappa = \dfrac{\det \mathrm{II}}{\det \mathrm{I}}$ ）

猜想：对任意曲面 $r=r(u,v)$，我们均有

$$\kappa = \frac{\det \mathrm{II}}{\det \mathrm{I}}$$

（注意观察：左边是几何量，故右边~~表达~~式必须在参数变换下不变!!!）

回忆：对任意平面曲线 $r=r(t)$，我们有

$$\kappa(t) = \frac{\langle \ddot{r}(t), n(t)\rangle}{|\dot{r}(t)|^2} = \frac{\mathrm{II}}{\mathrm{I}} = \frac{\det(\mathrm{II})}{\det(\mathrm{I})}$$

定理：对任意曲面 $r=r(u,v)$，高斯曲率

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}}.$$

证明：任取 $p=r(u_0,v_0)\in S$，我们来证：

$$\kappa(u_0,v_0) = \frac{\det \mathrm{II}(u_0,v_0)}{\det \mathrm{I}(u_0,v_0)} \qquad (*)$$

注意，根据高斯曲率定义：

$$K(u_0, v_0) = \lim_{\delta u, \delta v \to 0} \frac{\text{Area}(n(D))}{\text{Area}(r(D))}$$

$$= \frac{|(n_u \times n_v)(u_0, v_0)|}{|(r_u \times r_v)(u_0, v_0)|}$$

Step 1：定义（保向）参数变换

$\varphi : (\tilde{u}, \tilde{v}) \longrightarrow (u, v)$， 令 $\tilde{r} = r \circ \varphi$

$\tilde{D} \qquad D$

使得：(i) $\tilde{r}(0,0) = P$

(ii) $\underbrace{\tilde{r}_{\tilde{u}}}_{\text{切向量}}$, $\tilde{r}_{\tilde{v}}$ 在 $(0,0)$ 处垂直。

即 $\tilde{r}_{\tilde{u}}(0,0) \perp \tilde{r}_{\tilde{v}}(0,0)$

事实上，我们取一平移加线性变换即可。（行列式 >0）

证明留作习题。

step 2：(*) 右侧变为 $\dfrac{|\tilde{n}_{\tilde{u}} \times \tilde{n}_{\tilde{v}}(0,0)|}{|\tilde{r}_{\tilde{u}} \times \tilde{r}_{\tilde{v}}(0,0)|}$

(*) 右侧变为

$$\frac{\det(J\tilde{\mathrm{II}}J^T(0,0))}{\det(J\tilde{\mathrm{I}}J^T(0,0))} = \frac{\det(\tilde{\mathrm{II}}(0,0))}{\det(\tilde{\mathrm{I}}(0,0))},$$

其中 J 为 φ^{-1} 的 Jacobi 矩阵.

所以 (*) 化为

$$\frac{|\tilde{n}_{\tilde{u}} \times \tilde{n}_{\tilde{v}}(0,0)|}{|\tilde{r}_{\tilde{u}} \times \tilde{r}_{\tilde{v}}(0,0)|} = \frac{\det(\tilde{\mathrm{II}}(0,0))}{\det(\tilde{\mathrm{I}}(0,0))} \qquad (\tilde{*})$$

step 3: $\{\tilde{r}_{\tilde{u}}, \tilde{r}_{\tilde{v}}\}$ ~~$\{\tilde{n}_{\tilde{u}}, \tilde{n}_{\tilde{v}}\}$~~ 是切平面 T_pS 的基组量.

且由 Step 1 知

$$\left\{ \frac{\tilde{r}_{\tilde{u}}}{|\tilde{r}_{\tilde{u}}|}(0,0),\ \frac{\tilde{r}_{\tilde{v}}}{|\tilde{r}_{\tilde{v}}|}(0,0) \right\}$$ 是一组正交基.

故有等式.

$$\begin{cases} \tilde{n}_{\tilde{u}}(0,0) = \langle \tilde{n}_{\tilde{u}}(0,0), \tilde{r}_{\tilde{u}}(0,0)\rangle \dfrac{\tilde{r}_{\tilde{u}}(0,0)}{|\tilde{r}_{\tilde{u}}(0,0)|^2} + \\ \qquad \langle \tilde{n}_{\tilde{u}}(0,0), \tilde{r}_{\tilde{v}}(0,0)\rangle \dfrac{\tilde{r}_{\tilde{v}}(0,0)}{|\tilde{r}_{\tilde{v}}(0,0)|^2} \\ \tilde{n}_{\tilde{v}}(0,0) = \langle \tilde{n}_{\tilde{v}}(0,0), \tilde{r}_{\tilde{u}}(0,0)\rangle \dfrac{\tilde{r}_{\tilde{u}}(0,0)}{|\tilde{r}_{\tilde{u}}(0,0)|^2} + \end{cases}$$

$$\langle \tilde{n}_{\tilde{v}}(0,0),\ \tilde{r}_{\tilde{v}}(0,0)\rangle \frac{\tilde{r}_{\tilde{v}}(0,0)}{|\tilde{r}_{\tilde{v}}(0,0)|^2}$$

$$\Rightarrow\ |\tilde{n}_{\tilde{u}}(0,0) \times \tilde{n}_{\tilde{v}}(0,0)|$$

$$= \left[\frac{\langle \tilde{n}_{\tilde{u}}(0,0),\ \tilde{r}_{\tilde{u}}(0,0)\rangle \langle \tilde{n}_{\tilde{v}}(0,0),\ \tilde{r}_{\tilde{v}}(0,0)\rangle - \langle \tilde{n}_{\tilde{u}}(0,0),\ \tilde{r}_{\tilde{v}}(0,0)\rangle^2}{|\tilde{r}_{\tilde{u}}(0,0)|^2\,|\tilde{r}_{\tilde{v}}(0,0)|^2}\right]$$

$$\cdot\,|\tilde{r}_{\tilde{u}}(0,0) \times \tilde{r}_{\tilde{v}}(0,0)|$$

$$\Rightarrow\ (\tilde{※})\text{左側} = \frac{\det\left[\begin{pmatrix} \langle \tilde{r}_{\tilde{u}},\ \tilde{n}_{\tilde{u}}\rangle, & \langle \tilde{r}_{\tilde{u}},\ \tilde{n}_{\tilde{v}}\rangle \\ \langle \tilde{r}_{\tilde{v}},\ \tilde{n}_{\tilde{u}}\rangle, & \langle \tilde{r}_{\tilde{v}},\ \tilde{n}_{\tilde{v}}\rangle \end{pmatrix}(0,0)\right]}{\det\left[\begin{pmatrix} \langle \tilde{r}_{\tilde{u}},\ \tilde{r}_{\tilde{u}}\rangle & , \langle \tilde{r}_{\tilde{u}},\ \tilde{r}_{\tilde{v}}\rangle \\ \langle \tilde{r}_{\tilde{u}},\ \tilde{r}_{\tilde{v}}\rangle & , \langle \tilde{r}_{\tilde{v}},\ \tilde{r}_{\tilde{v}}\rangle \end{pmatrix}(0,0)\right]}$$

(注意: $\langle \tilde{r}_{\tilde{u}},\ \tilde{r}_{\tilde{v}}\rangle(0,0) = \langle \tilde{r}_{\tilde{v}},\ \tilde{r}_{\tilde{u}}\rangle(0,0) = 0$!)

$= (\tilde{※})$ 右側

#

§7. 高斯第一定理(1827)

考虑曲面 $z=f(x,y)$ 上一点P.

通过 $\{x,y,z\}$ 的坐标平移，设 $P=(0,0,0)$.

Taylor 展开，得

$$z = f'_x(0)\,x + f'_y(0)\,y + \frac{1}{2}f_{xx}(0)\,x^2 + f_{xy}(0)\,xy + \frac{1}{2}f_{yy}(0)\,y^2 + \text{高阶项}$$

$$\widetilde{n}_p = (-f_x(0), -f_y(0), 1)$$

取新的 xy-平面为 T_pS，则法向为 $(0,0,1)$

即在新坐标下面，$f'_x(0) = f'_y(0) = 0$. （习题）

[即找可逆阵 $A = \begin{pmatrix} B & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 1 \end{pmatrix}$. s.t. $\widetilde{n}_p \cdot A = (0,0,1)$].

故考虑

$$z = \frac{1}{2}f_{xx}(0)\,x^2 + f_{xy}(0)\,xy + \frac{1}{2}f_{yy}(0)\,y^2 + \text{高阶项}.$$

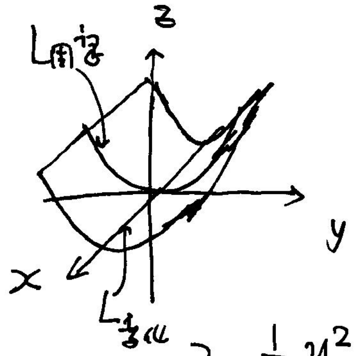
而通过选取新的坐标系，使得
（习题）

$$f_{xy}(0)=0, \quad \text{即得}$$

$$z = \frac{1}{2} f_{xx}(0)x^2 + \frac{1}{2} f_{yy}(0)y^2 + \text{高阶项}.$$

注意此时：
$$K(p) = \frac{f_{xx}(0) f_{yy}(0) - \cancelto{0}{f_{xy}^2(0)}}{(1+\cancelto{0}{f_x^2(0)}+\cancelto{0}{f_y^2(0)})^2}$$

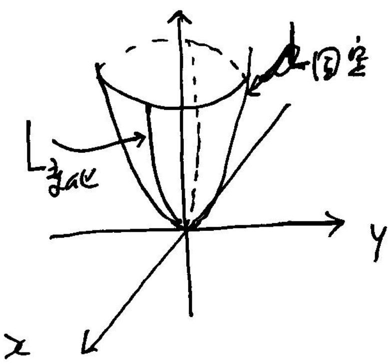
$$= f_{xx}(0)\cdot f_{yy}(0).$$



$z = \frac{1}{2}y^2$

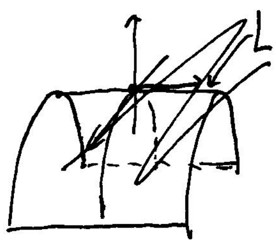
（抛物柱面）

例子：

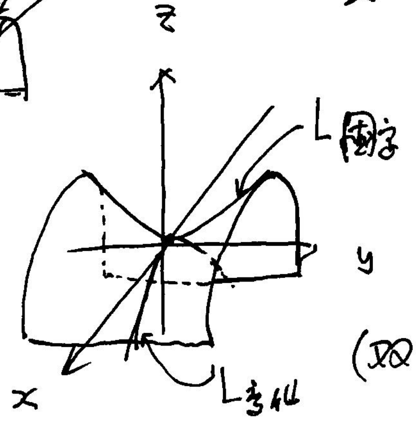


$z = \frac{1}{2}(x^2+y^2)$

（椭圆抛物面）



$z=$



$z = \frac{1}{2}(\cancel{x^2}\,{}^2+y^2)$

（双曲抛物面）

考察：(i) 由 xz 平面交 S 得曲线 得
$(y=0)$

$$z = \frac{1}{2} f_{xx}(0)x^2 + \text{高阶项（关于}x\text{）}$$

这平面曲线的曲率为 $|f_{xx}(0)|$ （习题）

(ii) 同理，由 yz-平面（$x=0$）截 S 得到的曲线（平面）

其曲率为 $|f_{yy}(0)|$

(iii) 更一般地，考虑过 法向 $(0,0,1)$ 及切向

$$\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\in T_pS=\mathrm{span}\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\}$$

的平面截 S 得平面曲线，其曲率为 截线为

$$z=\frac{1}{2}(f_{xx}(0)\cos^2\theta+f_{yy}(0)\sin^2\theta)\cdot r^2+O(r^3)$$

故其曲率为

$$|f_{xx}(0)\cdot\cos^2\theta+f_{yy}(0)\cdot\sin^2\theta|$$

这里更自然地引入有向曲率，即去掉记绝对值，则高斯观察到以下几何现象：

(i) 过法向及切平面中一切方向的平面截割曲面的平面曲线的有向曲率介于 $\min\{f_{xx}(0),f_{yy}(0)\}$ 与 $\max\{f_{xx}(0),f_{yy}(0)\}$ 之间。

且有

(ii) $K(0)=\min\{f_{xx}(0),f_{yy}(0)\}\cdot\max\{f_{xx}(0),f_{yy}(0)\}=f_{xx}(0)\cdot f_{yy}(0)$.

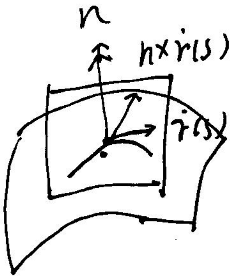
问题：如何推广到一般情形？

关键概念是"法曲率":

令 $p\in r(u(s), v(s))$ 为曲面上以弧长参数的曲线。

$$\dot r(s) = r_u \frac{du}{ds} + r_v \frac{dv}{ds} \in T_pS$$

$$\ddot r(s) = r_{uu}\left[\left(\frac{du}{ds}\right)^2\right] + 2 r_{uv}\left[\frac{du}{ds}\,\frac{dv}{ds}\right] + r_{vv}\left[\left(\frac{dv}{ds}\right)^2\right] + r_u\left[\frac{d^2u}{ds^2}\right] + r_v\left[\frac{d^2v}{ds^2}\right]$$



有直和分解: $T_p\mathbb{R}^3 = T_pS \oplus \mathbb{R}\{n_p\}$

$$\ddot r(s) = \underbrace{\qquad\qquad}_{\langle \ddot r(s),\, n\times \dot r(s)\rangle\, n\times\dot r(s)} + \langle \ddot r(s), n\rangle n$$

注意 $\{\dot r(s), n\times\dot r(s), n\}$ 构成 $T_p\mathbb{R}^3$ 正交标架, $\{\dot r(s), n\times\dot r(s)\}$ 是 T_pS 上正交标架

定义 法曲率(弧长参数)

$$k_n(s) \triangleq \langle \ddot r(s), n\rangle$$

「 $k_g(s) \triangleq \langle \ddot r(s), n\times\dot r(s)\rangle$ 测地曲率

故有 $k^2 = k_g^2 + k_n^2$, 其中 k 为(空间)曲线的曲率, 即 $|\ddot r(s)|$

我们将在以后研究 k_g 」

故 $$k_n(s) = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

注意到 $k_n(s)$ 只与 $\left\{\frac{du}{ds}, \frac{dv}{ds}\right\}$ 有关，而与 $\left\{\frac{d^2u}{ds^2}, \frac{d^2v}{ds^2}\right\}$ 无关.

一般地，设 $r(u(t), v(t))$ 为 S 上的参数曲线，我们

定义法曲率：

$$k_n(t) \triangleq \frac{\langle \ddot{r}(t), n\rangle}{\langle \dot{r}(t), \dot{r}(t)\rangle}$$

$$= \frac{w\,\mathrm{II}\,w^T}{w\,\mathrm{I}\,w^T}, \quad \text{其中}$$

$$\dot{r}(t) = \frac{du}{dt}\cdot r_u + \frac{dv}{dt} r_v = w\cdot\begin{pmatrix} r_u \\ r_v \end{pmatrix}$$

即 $w = \left(\frac{du}{dt}, \frac{dv}{dt}\right)$

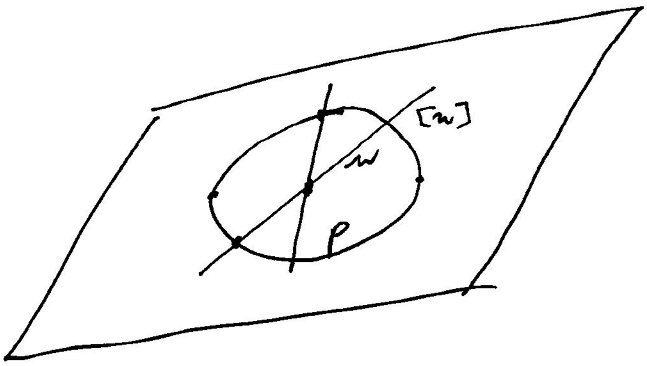
注意到：若令一曲线 $\overset{\tilde{r}(t)}{\overset{\shortparallel}{r(\tilde{u}(t), \tilde{v}(t))}}$ 与 $r(u(t), v(t))$ 在 P 点

有相同切线，即

$$\left(\frac{d\tilde{u}}{dt}, \frac{d\tilde{v}}{dt}\right) = \tilde{w} = \lambda w = \lambda\left(\frac{du}{dt}, \frac{dv}{dt}\right), \lambda \neq 0.$$

则 $\underset{k_n(w)}{\underset{\overset{\triangle}{\shortparallel}}{k_n(r(t))}} = \underset{k_n(\tilde{w})}{\underset{\shortparallel}{k_n(\tilde{r}(t))}}$，~~故 k_n 是~~

T_pS 上过点 p 的直线的集合是 $S^1/_{\{(x,-x)\}} \cong S^1$ 同胚



「例子：$S^1 \xrightarrow{\varphi} S^1$

$x \longmapsto x^2$

则 $\varphi^{-1}(x) = \{-x, x\}$」

故 $k_n : S^1 \longrightarrow \mathbb{R}^1$

$$[w] \longmapsto k_n(w)$$

显然 k_n 是连续函数。

$$\underbrace{S^1}_{\text{连通}}\ 紧集 \Rightarrow k_n(S^1) = \begin{cases} k \in \mathbb{R}^1 \\ [k_1, k_2] \in \mathbb{R}^1 \\ k_1 \neq k_2 \end{cases}$$

例：$z = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 +$ 高阶项，$k_1 \leq k_2$

$= f(x, y)$

则 $w = (\cos\theta, \sin\theta) \in T_pS$ $\qquad p = (0,0,0)$

$$\{\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\}$$

$k_n(w) = k_1\cos^2\theta + k_2\sin^2\theta$,

$\min\{f_{xx}(0), f_{yy}(0)\} = k_1$, $\quad \max\{f_{xx}(0), f_{yy}(0)\} = k_2$

故：法曲率 (k_n) 是高斯在圆柱形春季的轴截曲线曲率的合理推广.

<u>猜想</u>： $r=r(u,v)$ 一般曲面，则有

$$K(p)=k_1(p)\cdot k_2(p),$$ 其中

$k_1(p)$ ($k_2(p)$) 是过 p 点曲线的法曲率的最小值（最大值）.

我们称 k_1, k_2 为 p 点的主曲率。

克服问题的关键点在于理解：<u>高斯映射的切映射</u>：

$$(dG)_p:\ T_p(S)\longrightarrow T_{n(p)}(S^2)=T_p(S)$$

$$\lambda r_u+\mu r_v \longmapsto \lambda n_u+\mu n_v,\quad \forall \lambda,\mu\in R$$

$$\begin{cases} n_u = a r_u + b r_v \\ n_v = c r_u + d r_v \end{cases}$$

$$\Rightarrow \begin{cases} \langle r_u, n_u\rangle = a\langle r_u, r_u\rangle + b\langle r_u, r_v\rangle \\ \langle r_v, n_u\rangle = a\langle r_v, r_u\rangle + b\langle r_v, r_v\rangle \end{cases}$$

$$\begin{cases} \langle r_u, n_v\rangle = c\langle r_u, r_u\rangle + d\langle r_u, r_v\rangle \\ \langle r_v, n_v\rangle = c\langle r_v, r_u\rangle + d\langle r_v, r_v\rangle \end{cases}$$

故得

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = -\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

即 $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\mathrm{II}\cdot\mathrm{I}^{-1}$$

故得:

命题: 高斯映射在 p 点的切映射

$(dG)_p$ 在基 $\{r_u, r_v\}$ 下的表示式为

$$-\mathrm{II}(p)\,\mathrm{I}(p)^{-1}$$

定义: $-(dG)_p: T_pS \to T_pS$ Weingarten 映射.

$$\overset{\shortparallel}{W(p)} \quad \overset{基}{\{r_u, r_v\}} \mapsto \overset{基}{\{r_u, r_v\}}\left(\mathrm{II}\,\mathrm{I}^{-1}\right)$$

记 ~~$A=\mathrm{II}\,\mathrm{I}^{-1}$~~. 则有以下命题:

(1) $\cancel{\det A} = \det(\mathrm{II}\,\mathrm{I}^{-1}) = \det\mathrm{II}/\det\mathrm{I} = K$!

(2) ~~$A^T = A\,(\mathrm{II}\,\mathrm{I}^{-1})^T = \mathrm{I}$~~

$\varphi: (\tilde{u}, \tilde{v}) \longrightarrow (u, v)$ 为$\underbrace{\text{参数}}_{\text{同向}}$变换, 则

$\{r_{\tilde{u}}, r_{\tilde{v}}\} = \{r_u, r_v\}\cdot J$ 为 T_pS 的另一组基.

则

$\mathcal{W}(p)$ 在新基 $\{r_{\tilde{u}}, r_{\tilde{v}}\}$ 的表出式应为

$$J^{-1}(\mathrm{II}\,\mathrm{I}^{-1})J$$

即 $\mathcal{W}(p)\{r_{\tilde{u}}, r_{\tilde{v}}\} = \{r_{\tilde{u}}, r_{\tilde{v}}\}\left(J^{-1}\mathrm{II}\,\mathrm{I}^{-1}J\right)$

但 $J^{-1}(\mathrm{II}\,\mathrm{I}^{-1})J = \left(J^{-1}\mathrm{II}\,J^{T}\right)(J^{-T}\mathrm{I}^{-1}J)$ $\quad (J^{-1}\mathrm{I}J^{T})^{-1}$

$$= \tilde{\mathrm{II}}\cdot\tilde{\mathrm{I}}^{-1}$$

故有 $\mathcal{W}(p)\{r_{\tilde{u}}, r_{\tilde{v}}\} = \{r_{\tilde{u}}, r_{\tilde{v}}\}\left(\tilde{\mathrm{II}}, \tilde{\mathrm{I}}^{-1}\right)$

即 Weingarten 映射在参数变换下的不变性.

由(2), 我们可以有如下结论:

我们可以取 $\{\tilde{u}, \tilde{v}\}$, 使得 $\tilde{\mathrm{I}}(p) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, 即

$\{r_{\tilde{u}}(p), r_{\tilde{v}}(p)\}$ 为正交标架. (习题)

则 $\mathcal{W}(p)\{r_{\tilde{u}}(p), r_{\tilde{v}}(p)\} = \{r_{\tilde{u}}(p), r_{\tilde{v}}(p)\}\cdot\tilde{\mathrm{II}}(p)$

<u>高斯第一定理</u>: $r = r(u,v)$ 曲面, 则

$$\boxed{K(p) = k_1(p)\cdot k_2(p)}$$ 高斯曲率为两主曲率之积.

证明: 设 $\{e_1 = r_u(p),\ e_2 = r_v(p)\}$ 为 T_pS 的正交标架。73

Step 1: $w = \cos\theta\, e_1 + \sin\theta\, e_2 \in T_pS$ 为单位切向量场

则

$$k_n(w) = \frac{w\,\mathrm{II}\,w^T}{w\,\mathrm{I}\,w^T} = w\cdot\mathrm{II}\cdot w^T$$

$$= \langle W(p)\cdot(w),\ w\rangle$$

Step 2: $W(p)\{e_1, e_2\} = \{e_1, e_2\}\,\mathrm{II}(p)$

$$\mathrm{II}(p) = \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad \mathrm{II}(p)^T = \mathrm{II}(p)$$

$\Rightarrow$ 特征多项式

$$t^2 - \overset{tr(\mathrm{II})}{\overset{\parallel}{(L+N)}}\,t + \overset{det(\mathrm{II})}{\overset{\parallel}{(LN-M^2)}} = 0$$

$$\Delta = (L+N)^2 - 4(LN-M^2)$$

$$= (L-N)^2 + 4M^2 \geqslant 0$$

$(\Delta = 0 \Leftrightarrow L=N,\ M=0)$

$\Rightarrow$ $W(p)$ 有两实特征根 $\lambda_1 \leqslant \lambda_2$

且 若 $\lambda_1 \neq \lambda_2$, 则 对应特征向量 v_1, v_2 有 互相垂直

$$\langle v_1, v_2\rangle = 0 \qquad (\text{习题})$$

故总可取两互相垂直单位向量 v_1, v_2，使得

$$W(p)(v_1) = \lambda_1 v_1, \quad W(p)(v_2) = \lambda_2 v_2$$

Step 3: 以 $\{v_1, v_2\}$ 为新正交标架. 有

$$w = \cos(\varphi) v_1 + \sin(\varphi) v_2$$

$$k_n(w) = \langle W(p)(\cos(\varphi) v_1 + \sin(\varphi) v_2), \cos(\varphi) v_1 + \sin(\varphi) v_2 \rangle$$

$$= \cos^2(\varphi) \underbrace{\langle W(p)(v_1), v_1 \rangle}_{= \lambda_1 \langle v_1, v_1 \rangle = \lambda_1} + \sin^2(\varphi) \underbrace{\langle W(p)(v_2), v_2 \rangle}_{= \lambda_2}$$

故有：
$$\begin{cases} k_1 = \lambda_1, \\ k_2 = \lambda_2 \end{cases}$$

且 $\underbrace{\lambda_1 \cdot \lambda_2}_{= k_1 \cdot k_2} = \det(\mathrm{II}) = K$

#

命题：$r = r(u, v)$ 曲面在P点的主曲率为 k_1, k_2，则 r 在 P 点的二阶近似曲面为 $z = \frac{1}{2}(k_1 u^2 + k_2 v^2)$.

证明:

Step 1: 可取参数 (u,v) 使得

$\{r_u(p), r_v(p)\}$ 为单位正交基，且

$$W(p)(r_u(p)) = k_1(r_u(p)),\quad W(p)(r_v(p)) = k_2(r_v(p))$$

且 $\{r_u(p), r_v(p)\}$ 为单位正交主方向..

(可验) $\mathrm{I}(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{II}(p) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$.

Step 2. 取 $\mathbb{R}^3$ 在点 p 处旋转，使

$$r_u(p) = e_1,\quad r_v(p) = e_2,\quad n(p) = e_3$$

则

$$\begin{aligned}
\Delta r &= r(u_0+\Delta u, v_0+\Delta v) - r(u_0, v_0) \\
&= r_u(u_0,v_0)\Delta u + r_v(u_0,v_0)\Delta v + \\
&\quad \frac{1}{2}\left(r_{uu}(u_0,v_0)(\Delta u)^2 + 2r_{uv}(u_0,v_0)(\Delta u)(\Delta v) + r_{vv}(u_0,v_0)(\Delta v)^2\right) \\
&\quad + o\left[(\Delta u)^2 + (\Delta v)^2\right] \\
&= \left(\Delta u + o\left(\sqrt{(\Delta u)^2+(\Delta v)^2}\right)\right)e_1 + \left(\Delta v + o\left(\sqrt{(\Delta u)^2+(\Delta v)^2}\right)\right)e_2 + \\
&\quad \frac{1}{2}\left(k_1(\Delta u)^2 + k_2(\Delta v)^2 + o((\Delta u)^2+(\Delta v)^2)\right)e_3
\end{aligned}$$

即得所证. #